stochastic processes and their

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Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading ☆

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Abstract

Motivated by the Kyle–Back model of "insider trading", we consider certain classes of linear transformations of two independent Brownian motions and study their canonical decomposition, i.e., their Doob–Meyer decomposition as semimartingales in their own filtration. In particular we characterize those transformations which generate again a Brownian motion. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider two independent Brownian motions $(W_t)_{t\geqslant 0}$ and $(\tilde{W}_t)_{t\geqslant 0}$. We study solutions X of stochastic differential equations

$$dX_t = dW_t + Y_t\,dt \tag{1}$$

driven by W, where the drift Y depends linearly on X and $\tilde{W}$. Our purpose is to derive the canonical decomposition of X, i.e., the Doob–Meyer decomposition of X viewed

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as a semimartingale in its own filtration $(\mathscr{F}_t^X)$. In particular, we want to characterize those cases where X is again a Brownian motion.

This investigation was motivated by the following example. Consider the Brownian bridge from 0 to $\tilde{W}_1$ defined by (1), but with an anticipating drift given by

$$Y_t = \frac{\tilde{W}_1 - X_t}{1-t}. \tag{2}$$

The process X is a new Brownian motion such that $X_1 = \tilde{W}_1$. This example plays a crucial role in the Kyle–Back model of "insider trading" (see Kyle, 1985; Back, 1992). The "insider" knows in advance the final value $\tilde{W}_1$. He applies the drift (2) in order to modify the original Brownian motion W in such a way that (i) the resulting process X ends up in $\tilde{W}_1$, and (ii) the distribution of the process remains unchanged, i.e., X is again a Brownian motion. Condition (i) guarantees that the strategy maximizes the insider's expected gain; cf. Back (1992). Condition (ii) means that the strategy (Y_t) of the insider is "inconspicuous", and this corresponds to the notion of equilibrium as defined in Back (1992).

Let us now modify the example as follows. Suppose that the "insider" cannot anticipate the final value $\tilde{W}_1$ from the beginning. Instead, his "insider information" consists in observing the second Brownian motion $\tilde{W}$. This suggests to replace the anticipating drift (2) by the adapted drift

$$Y_t = \frac{\tilde{W}_t - X_t}{1-t}. \tag{3}$$

The resulting process X converges again to $\tilde{W}_1$. But its distribution has changed: X is no longer a Brownian motion. Thus, the strategy (Y_t) is no longer inconspicuous. In Section 2 we give a precise description of this effect by computing the drift which arises in the canonical decomposition of X as a semimartingale in its own filtration.

The preceding example suggests to study a more general class of processes of the form (1) where the drift is given as a time-dependent linear function in X and $\tilde{W}$, i.e.,

$$Y_t = f(t)\tilde{W}_t + h(t)X_t \tag{4}$$

for functions f and h in $C^1(0,1)$ satisfying some mild integrability conditions. In Section 3 we derive the canonical decomposition of X in its own filtration. To this end, we consider the Radon–Nikodym density D of the law of X with respect to that of W. We express D in terms of $(W, \tilde{W})$, and we compute in closed form $E[D \mid W]$, its conditional expectation with respect to W. Applying one more time the Girsanov transformation we obtain the canonical decomposition of X. In order to formulate the result, we introduce the fundamental solution $\Psi(t)$ of the Sturm–Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t), \tag{5}$$

with boundary conditions $\Psi(0) = 0$ and $\Psi'(0+) = 1$. The canonical decomposition of X is given by

$$X_t = B_t + \int_0^t (f(u)k_u(X_v;\ v \leqslant u) + h(u)X_u)\,\mathrm{d}u,$$

where (B_t) is an $(\mathscr{F}_t^X)$-Brownian motion and the functional k_u is given by

$$k_u(X_s;\ s\leqslant u) = \frac{1}{\Psi'(u)} \int_0^u \Psi(v)(f(v)\mathrm{d}X_v - f(v)h(v)X_v\,\mathrm{d}v).$$

An alternative method consists in applying the stochastic filtering theory for Gaussian processes. This is explained in Section 4. There we consider stochastic differential equations (1) where the drift is given by an adapted linear transformation of X and $\tilde{W}$, i.e.,

$$\mathrm{d}X_t = \mathrm{d}W_t + \left(\int_0^t F(t,u)\mathrm{d}\tilde{W}_u + \int_0^t H(t,u)\mathrm{d}X_u \right)\mathrm{d}t, \tag{6}$$

with some square-integrable Volterra kernels F and H; see the definition in Section 3.2. Theorem 4.1 shows that the canonical decomposition of X is of the form

$$X_t = B_t + \int_0^t \left(\int_0^s G_F(s,u)\mathrm{d}B_u + \int_0^s H(s,u)\mathrm{d}X_u \right)\mathrm{d}s, \tag{7}$$

where (B_t) is a standard Brownian motion with respect to $(\mathscr{F}_t^X)$ and G_F is the square-integrable Volterra kernel determined by the equation

$$\int_0^s F(s,u)F(t,u)\mathrm{d}u = G_F(t,s) + \int_0^s G_F(s,u)G_F(t,u)\mathrm{d}u. \tag{8}$$

In the special cases considered in Section 3, the kernel G_F can be identified in terms of the solution of a Sturm–Liouville equation.

In Section 5.1 we return to the discussion of properties (i) and (ii) which appeared in our initial example (2). In the general context of Eq. (6) where the drift Y is a linear functional of the past of X and $\tilde{W}$, we characterize those cases which satisfy condition (ii), i.e., the resulting process X is again a Brownian motion. The criterion is that

$$H(t,u) = -G_F(t,u),$$

where G_F is given by (8). But it is not possible to obtain at the same time condition (i), i.e., to tie such a Brownian motion X to the endpoint of the Brownian motion $\tilde{W}$. In fact, the simple argument of Proposition 5.1 shows that there is no drift (Y_t) adapted to $(W,\tilde{W})$ such that the solution X of (1) is a Brownian motion with endpoint $X_1=\tilde{W}_1$. In the context of the financial interpretation where $\tilde{W}$ describes the information available to the "insider", we have thus shown that there exists no strategy (Y_t) which satisfies both equilibrium conditions (i) and (ii). The situation may change if $\tilde{W}$ is replaced by a Gaussian martingale $S=(S_t)$ with $S_1=\tilde{W}_1$, but with a slower increase of the variance; see Back and Pedersen (1998) and Wu (1999). Note that the initial example (2) is of this form, with $S_t \equiv \tilde{W}_1$. But our negative result in the case $S_t \equiv \tilde{W}_t$ shows that, in general, we may have to relax the notion of equilibrium. For example, we could look at strategies which minimize some functional depending both on the deviation between X_1 and $\tilde{W}_1$ and on the deviation between Wiener measure $\mathbb{P}$ and the law $\mathbb{Q}$ of X, e.g., in terms of the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. Such extensions of the equilibrium concept will be discussed elsewhere. In particular, they will involve the canonical decomposition of X as a semimartingale in its own filtration. The present paper provides such decompositions for certain classes of linear strategies.

In Section 5.2 we point out the following connection of our decomposition results to an enlargement of filtration. First we note that a Brownian motion X, given as the solution of Eq. (7) with $H = -G_F$, can be expressed directly in terms of W and $\tilde{W}$:

$$dX_t = dW_t + \left\{ \int_0^t L_F(t,u)dW_u + \int_0^t \left(F(t,u) + \int_u^t L_F(t,v)F(v,u)dv \right) d\tilde{W}_u \right\} dt, \tag{9}$$

where L_F denotes the resolvent kernel of G_F; see (68). In the special case $F(t,u)=f(t)$, this can be reduced to

$$W_t = X_t - \int_0^t \frac{f(u)}{\Psi'(u)} \left(\int_0^u \Psi'(v)d\tilde{W}_v - \int_0^u f(v)\Psi(v)dW_v \right) du,$$

where $\Psi(t)$ is the solution of the Sturm–Liouville equation (5). This representation of W in terms of the Brownian motion X can be viewed, after time reversal, as the decomposition of a Brownian motion in some enlarged Gaussian filtration.

2. A bridge between two Brownian motions

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $(W_t)_{0\leqslant t\leqslant 1}$ be a standard Brownian motion with respect to its canonical filtration $(\mathscr{F}_t^W)_{0\leqslant t\leqslant 1}$. Now let $(\tilde{W}_t)_{0\leqslant t\leqslant 1}$ be another standard Brownian motion on the same probability space which is independent of $(W_t)_{0\leqslant t\leqslant 1}$, and denote by $(\mathscr{F}_t)_{0\leqslant t\leqslant 1}$ the filtration generated by these two Brownian motions.

We know that the solution $(\tilde{X}_t)_{0\leqslant t\leqslant 1}$ of the stochastic differential equation

$$d\tilde{X}_t = dW_t + \frac{\tilde{W}_1 - \tilde{X}_t}{1-t} dt, \tag{10}$$

with initial value $\tilde{X}_0 = 0$, is a standard Brownian motion which converges to the final value $\tilde{W}_1$ (cf., for example, Jeulin and Yor, 1979). Now we look at the process $(X_t)_{0\leqslant t\leqslant 1}$ starting in $X_0 = 0$ which is defined by the stochastic differential equation

$$dX_t = dW_t + \frac{\tilde{W}_t - X_t}{1-t} dt. \tag{11}$$

Clearly, for any $t \in [0,1]$, X_t is normally distributed, and $\langle X \rangle_t = t$. The following lemma shows that X_t approaches $\tilde{W}_1$ as $t \to 1$. However, we will see that $(X_t)_{0\leqslant t\leqslant 1}$ is no longer a Brownian motion.

Lemma 2.1. *$X_t \to \tilde{W}_1$ $\mathbb{P}$-a.s. as $t \to 1$.*

Proof. The explicit solution of (11) is given by

$$X_t = (1-t) \int_0^t \frac{\tilde{W}_s}{(1-s)^2} ds + (1-t) \int_0^t \frac{1}{(1-s)} dW_s. \tag{12}$$

The first term approaches $\tilde{W}_1$ and the second goes to 0 as $t \to 1$, and this implies the result. Alternatively, we could note that the process $2^{-1/2}(X - \tilde{W})$ satisfies the equation of a Brownian bridge tied down to the final value 0. □

Lemma 2.2. *For* $0 \leqslant s \leqslant t \leqslant 1$, *we have*

$$E[X_t\tilde{W}_t] = t + (1-t)\log(1-t), \tag{13}$$

and the covariance function of X *is given by*

$$E[X_sX_t] = s + 2s(1-t) + (2-s-t)\log(1-s). \tag{14}$$

Proof. Applying the integration by parts formula to the first integral in (12), the solution of (11) is given by

$$X_t = (1-t)\int_0^t \frac{\mathrm{d}W_s - \mathrm{d}\tilde{W}_s}{1-s} + \tilde{W}_t.$$

Since (W_t) and $(\tilde{W}_t)$ are independent, we establish

$$\begin{aligned} E[X_t\tilde{W}_t] &= t + (1-t)E\int_0^t \frac{\tilde{W}_t(\mathrm{d}W_s - \mathrm{d}\tilde{W}_s)}{1-s} \\ &= t + (1-t)\log(1-t), \end{aligned}$$

and

$$\begin{aligned} E[X_sX_t] &= E[\tilde{W}_s\tilde{W}_t] + (1-t)(1-s)E\left[\left(\int_0^s \frac{\mathrm{d}W_u - \mathrm{d}\tilde{W}_u}{1-u}\right)^2\right] \\ &\quad +(1-s)E\left[\tilde{W}_t\int_0^s \frac{\mathrm{d}W_u - \mathrm{d}\tilde{W}_u}{1-u}\right] + (1-t)E\left[\tilde{W}_s\int_0^t \frac{\mathrm{d}W_u - \mathrm{d}\tilde{W}_u}{1-u}\right] \\ &= s + 2s(1-t) + (2-s-t)\log(1-s). \qquad \square \end{aligned}$$

This lemma shows that $(X_t)_{0\leqslant t\leqslant 1}$ is not a Brownian motion, since its covariance function differs from $t \wedge s$. But it is a semimartingale with respect to $(\mathscr{F}_t)_{0\leqslant t\leqslant 1}$; this follows from the last sentence in the proof of Lemma 2.1 and the fact that the Brownian bridge has this property; see, e.g., Protter (1992, p. 244). Therefore, it is obviously a semimartingale relative to its natural filtration. A natural question is: what is the explicit form of its canonical decomposition? That is the problem we want to discuss in this section.

Lemma 2.3. *Suppose that the process* $(X_t)_{t\geqslant 0}$ *is given by*

$$X_t = W_t + \int_0^t Y_u \,\mathrm{d}u,$$

with a Brownian motion $(W_t)_{t\geqslant 0}$ *adapted to a filtration* $(\mathscr{F}_t)_{t\geqslant 0}$ *and an* $(\mathscr{F}_t)$*-adapted process* $(Y_t)_{t\geqslant 0}$ *satisfying* $\int_0^t E|Y_u|\mathrm{d}u < \infty$ *for all* t.

(1) *The canonical decomposition of* X *in its natural filtration* $(\mathscr{F}_t^X)$ *is given by*

$$X_t = B_t + \int_0^t E[Y_u|\mathscr{F}_u^X]\mathrm{d}u, \tag{15}$$

where the process B *defined by* (15) *is an* $(\mathscr{F}_t^X)$*-Brownian motion, which is often called the innovation process of* X. *In particular,* (X_t) *is a Brownian motion if and only if*

$$E[Y_u|\mathscr{F}_u^X] = 0, \quad \mathrm{d}\mathbb{P}\times\mathrm{d}u\text{-a.s.}$$

(2) *Furthermore, if the function* $s \mapsto Y_s$ *is* L^1*-continuous on* $(0,\infty)$ *and if* $(X_t)_{t\geqslant 0}$ *is a Gaussian process, then* $(X_t)_{t\geqslant 0}$ *is a Brownian motion if and only if*

$$E(X_sY_t) = 0, \tag{16}$$

for all $0 < s \leqslant t$.

Proof. (1) The first part of the first assertion can be found in Hida and Hitsuda (1993) or Liptser and Shiryaev (1977, Theorem 7.12). For completeness, here is the well-known argument: for $s < t$,

$$E[X_t - X_s|\mathscr{F}_s^X] = E\left[\int_s^t Y_u \,\mathrm{d}u|\mathscr{F}_s^X\right] = E\left[\int_s^t E[Y_u|\mathscr{F}_u^X]\mathrm{d}u|\mathscr{F}_s^X\right].$$

Hence,

$$B_t := X_t - \int_0^t E[Y_u|\mathscr{F}_u^X]\mathrm{d}u$$

is an $(\mathscr{F}_t^X)$-martingale; since $\langle X\rangle_t = t$, (B_t) is an $(\mathscr{F}_t^X)$-Brownian motion. The second part is immediate from the uniqueness of the canonical decomposition of X in $(\mathscr{F}_t^X)$.

(2) For the second assertion, suppose that (X_t) is a Brownian motion. Then $E(X_sX_t) = s$ and, therefore,

$$\int_0^t E(Y_uX_s)\mathrm{d}u = s - E(X_sW_t) = -\int_0^s E(Y_uW_s)\mathrm{d}u.$$

Taking derivatives with respect to t on both sides, we obtain (16). Conversely, from Stricker (1983, Theorem 1) we know that if X is a Gaussian semimartingale, then its canonical decomposition is jointly Gaussian, i.e., $(X_t, \int_0^t E[Y_u|\mathscr{F}_u^X]\mathrm{d}u)_{t\geqslant 0}$ is a Gaussian process. Since $s \mapsto Y_s$ is L^1-continuous, $(X_t, E[Y_t|\mathscr{F}_t^X])_{t\geqslant 0}$ is Gaussian; cf. Stricker (1983, Lemma 2). From (16) we have $E[Y_t|\mathscr{F}_t^X] = 0$, and so X is a Brownian motion due to (15). $\square$

Corollary 2.1. *Let the process* $(X_t)_{0\leqslant t\leqslant 1}$ *satisfy* (11). *Then the process B, defined as*

$$B_t := X_t - \int_0^t \frac{E[\tilde{W}_u|\mathscr{F}_u^X] - X_u}{1-u}\mathrm{d}u, \tag{17}$$

is a Brownian motion relative to $(\mathscr{F}_t^X)_{0\leqslant t\leqslant 1}$.

Proof. Set

$$Y_u = \frac{\tilde{W}_u - X_u}{1-u},$$

then from the first assertion in Lemma 2.3, we obtain the required result. $\square$

Therefore, we have only to compute the conditional expectation of $\tilde{W}_t$ relative to $\mathscr{F}_t^X$.

Lemma 2.4. *Set* $A := \frac{1}{2}(1+\sqrt{5})$ *and* $B := \frac{1}{2}(1-\sqrt{5})$. *Then for* $0 \leqslant t < 1$,

$$E[\tilde{W}_t|\mathscr{F}_t^X] = \int_0^t \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-t)^{-B} - B(1-t)^{-A}}\mathrm{d}X_s + X_t. \tag{18}$$

Proof. Due to (17) and Theorem 9.4.1 in Kallianpur (1980), we may assume that the conditional expectation of $\tilde{W}_t$ with respect to the σ-algebra $\mathscr{F}_t^X$ is of the form

$$E[\tilde{W}_t|\mathscr{F}_t^X] = \int_0^t a(t,u)\,\mathrm{d}X_u.$$

Applying the projection property of the conditional expectation

$$E[X_s(\tilde{W}_t - E[\tilde{W}_t|\mathscr{F}_t^X])] = 0,$$

for all $0 \leqslant s \leqslant t < 1$, as well as the martingale property we obtain

$$E(X_s\tilde{W}_s) - E(X_sX_t)a(t,t) = -\int_0^t E(X_sX_u)\frac{\partial a(t,u)}{\partial u}\,\mathrm{d}u.$$

Using (13), (14) and computing explicitly the left-hand side (LHS) and the right-hand side (RHS) in this equation, we get

$$\mathrm{LHS} = s + (1-s)\log(1-s) - (s + 2s(1-t) + (2-s-t)\log(1-s))a(t,t).$$

$$\mathrm{RHS} = -\int_0^s (u + 2u(1-s) + (2-s-u)\log(1-u))\frac{\partial a(t,u)}{\partial u}\,\mathrm{d}u$$
$$-\int_s^t (s + 2s(1-u) + (2-s-u)\log(1-s))\frac{\partial a(t,u)}{\partial u}\,\mathrm{d}u.$$

Taking the second derivatives with respect to s on both sides implies

$$\frac{1}{1-s} - \frac{(t-s)}{(1-s)^2}a(t,t) = \frac{\partial a(t,s)}{\partial s} - \int_s^t \frac{(u-s)}{(1-s)^2}\frac{\partial a(t,u)}{\partial u}\,\mathrm{d}u.$$

Multiplication of both sides with $(1-s)^2$ leads to

$$1 - s - (t-s)a(t,t) = (1-s)^2\frac{\partial a(t,s)}{\partial s} - \int_s^t (u-s)\frac{\partial a(t,u)}{\partial u}\,\mathrm{d}u,$$

and this implies

$$1 - s = (1-s)^2\frac{\partial a(t,s)}{\partial s} + \int_s^t a(t,u)\mathrm{d}u.$$

Taking further derivatives with respect to s on both sides we get

$$(1-s)^2\frac{\partial^2 a(t,s)}{\partial s^2} - 2(1-s)\frac{\partial a(t,s)}{\partial s} - a(t,s) + 1 = 0.$$

The solution of this differential equation is given by

$$a(t,s) = c(t)(1-s)^{-A} + d(t)(1-s)^{-B} + 1.$$

Substituting this equation in RHS and comparing the coefficients of s, $\log(1-s)$ and $s\log(1-s)$ in LHS and RHS, we derive the desired result. □

Therefore, Corollary 2.1 allows us to conclude:

Proposition 2.1. *The canonical decomposition of X is given by*

$$X_t = B_t + \int_0^t\int_0^u \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-u)^A - B(1-u)^B}\,\mathrm{d}X_s\,\mathrm{d}u, \tag{19}$$

for $0 \leqslant t < 1$.

Remark 2.1. Let $(X_t)_{0\leqslant t\leqslant 1}$, a centered Gaussian semimartingale, be of the form

$$X_t = W_t + \int_0^t Y_s\,\mathrm{d}s.$$

A result of Hitsuda (1994) states that the conditional expectation $E[Y_s|\mathscr{F}_s^X]$ is equal to the (orthogonal) projection of Y_s to the space $H_s(X)$, which is the L^2-closure of the set that consists of all stochastic integrals of the form: $\int_0^s f(u)\mathrm{d}X_u$ with bounded Borel function f. In our situation (11), the explicit form of this projection is given by (19).

3. Canonical decompositions, Sturm–Liouville equations, and Volterra representations

Now we consider the case where the process $(X_t)_{0\leqslant t<1}$ satisfies the stochastic differential equation:

$$\mathrm{d}X_t = \mathrm{d}W_t + (f(t)\tilde{W}_t + h(t)X_t)\mathrm{d}t, \tag{20}$$

with initial value $X_0 = 0$. The functions f and h are assumed to belong to $C^1(0,1)\cap\mathscr{A}(0,1)$, the space $\mathscr{A}(0,1)$ being defined as the set of all measurable functions ϕ with $\int_0^t s\phi^2(s)\mathrm{d}s<\infty$, for all $t<1$.

From Girsanov's transformation, the law of $(X_s\,;s\leqslant t)$, for any $t<1$, may be written in terms of that of $(W,\tilde{W})$ via the following formula:

$$E[F(X_s;\ s\leqslant t)] = E[F(W_s;\ s\leqslant t)\mathscr{E}_t], \tag{21}$$

where $F\geqslant 0$ is a measurable functional and

$$\varepsilon_t = \exp\left(\int_0^t (f(u)\tilde{W}_u + h(u)W_u)\mathrm{d}W_u - \frac{1}{2}\int_0^t (f(u)\tilde{W}_u + h(u)W_u)^2\mathrm{d}u\right).$$

Since $(\mathscr{E}_t)_{0\leqslant t<1}$ is a martingale with respect to $(\mathscr{F}_t)$, the natural filtration generated by $(W_t)_{0\leqslant t\leqslant 1}$ and $(\tilde{W}_t)_{0\leqslant t\leqslant 1}$, we know $\Lambda_t := E(\mathscr{E}_t|\mathscr{F}_t^W)$ is also a martingale with respect to $(\mathscr{F}_t^W)_{0\leqslant t<1}$. Once we have obtained in the next section a closed form of Λ_t, we shall apply again Girsanov's transformation to get the canonical decomposition of $(X_t)_{0\leqslant t<1}$.

3.1. Computation of Λ_t

Obviously, $\mathscr{E}_t$ may be decomposed as: $\mathscr{E}_t = \mathscr{E}_t^{(1)}\mathscr{E}_t^{(2)}$, where

$$\mathscr{E}_t^{(1)} := \exp\left\{\int_0^t h(u)W_u\,\mathrm{d}W_u - \frac{1}{2}\int_0^t h^2(u)W_u^2\,\mathrm{d}u\right\}$$

and

$$\mathscr{E}_t^{(2)} := \exp\left\{\int_0^t f(u)\tilde{W}_u\,\mathrm{d}W_u - \int_0^t f(u)h(u)W_u\tilde{W}_u\,\mathrm{d}u - \frac{1}{2}\int_0^t f^2(u)\tilde{W}_u^2\,\mathrm{d}u\right\},$$

so that

$$\Lambda_t = \mathscr{E}_t^{(1)} E[\mathscr{E}_t^{(2)} | \mathscr{F}_t^W].$$

Thus, in order to compute Λ_t, it will suffice to obtain a "robust" formula for the expectation

$$I_{\mu,\nu}(t) := E[\mathscr{E}_{\mu,\nu}(t)]$$

of the exponential

$$\mathscr{E}_{\mu,\nu}(t) = \exp\left(\int_0^t \tilde{W}_s \nu(\mathrm{d}s) - \frac{1}{2} \int_0^t \tilde{W}_s^2 \mu(\mathrm{d}s) \right),$$

where $\mu(\mathrm{d}s) := f^2(s)\mathrm{d}s$, and $\nu(\mathrm{d}s)$ is a generic signed measure. Later, we shall justify the replacement of $\nu(\mathrm{d}s)$ by the Gaussian "measure":

$$\nu_W(\mathrm{d}s) := f(s)\mathrm{d}W_s - f(s)h(s)W_s\,\mathrm{d}s.$$

In order to present the next "explicit" formula for $I_{\mu,\nu}(t)$, we need to introduce two fundamental solutions $\Phi(u)$ and $\Psi(u)$ of the Sturm–Liouville equation:

$$\phi''(\mathrm{d}u) = \mu(\mathrm{d}u)\phi(u),$$

relative to a measure μ, which are characterized by:

(i) $\Phi(u)$ is decreasing, and satisfies $\Phi(0) = 1$;
(ii) $\Psi(u)$ satisfies $\Psi(0) = 0$ and $\Psi'(0+) = 1$;

see, e.g., Revuz and Yor (1999, pp. 550–551). Since

$$\Psi(u) = \Phi(u) \int_0^u \frac{1}{\Phi^2(v)} \mathrm{d}v, \tag{22}$$

we get the important Wronskian's relation between Φ and Ψ:

$$\Phi(u)\Psi'(u) - \Psi(u)\Phi'(u) = 1. \tag{23}$$

Throughout this section, we discuss the case $\mu(\mathrm{d}s) = f^2(s)\,\mathrm{d}s$. Therefore, the corresponding Sturm–Liouville equation is given by

$$\phi''(u) = f^2(u)\phi(u). \tag{24}$$

Proposition 3.1. *The expectation $I_{\mu,\nu}(t) = E[\mathscr{E}_{\mu,\nu}(t)]$ is equal to*

$$\frac{1}{\sqrt{\Psi'(t)}} \exp\left\{ \frac{1}{2} \int_0^t \left(\int_u^t \frac{\Phi(s)}{\Phi(u)} \nu(\mathrm{d}s) \right)^2 \mathrm{d}u - \theta(t) \left(\int_0^t \Psi(s)\nu(\mathrm{d}s) \right)^2 \right\}, \tag{25}$$

where

$$\theta(t) := \Phi'(t)/(2\Psi'(t)). \tag{26}$$

Proof. The proof simply consists in pushing the computation made in Pitman and Yor (1982) a little further. Precisely, changing Wiener measure with the Radon–Nikodym density

$$Z_t^\mu := \exp\left\{ \frac{1}{2}(F(t)\tilde{W}_t^2 - \hat{F}(t)) - \frac{1}{2} \int_0^t \tilde{W}_s^2 \mu(\mathrm{d}s) \right\},$$

with $F(t)=\Phi'(t)/\Phi(t)$ and $\hat{F}(t)=\int_0^t F(s)\,\mathrm{d}s=\log\Phi(t)$, it follows that

$$I_{\mu,v}(t)=E^{\mu}\left[\exp\left\{-\frac{1}{2}(F(t)\tilde{W}_t^2-\hat{F}(t))+\int_0^t \tilde{W}_s v(\mathrm{d}s)\right\}\right],$$

where, under the new probability measure $\mathbb{P}^{\mu}$, the process $(\tilde{W}_t)_{0\leqslant t\leqslant 1}$ satisfies

$$\tilde{W}_t=B_t+\int_0^t F(s)\tilde{W}_s\,\mathrm{d}s,$$

with a $\mathbb{P}^{\mu}$-Brownian motion $(B_t)_{0\leqslant t\leqslant 1}$. Consequently, $(\tilde{W}_t)_{0\leqslant t\leqslant 1}$ is a centered Gaussian process under $\mathbb{P}^{\mu}$. Now we write

$$I_{\mu,v}(t)=\exp\left(\frac{1}{2}\hat{F}(t)\right)E^{\mu}\left[\exp\left(\int_0^t \tilde{W}_s v(\mathrm{d}s)+\sqrt{-F(t)}N\tilde{W}_t\right)\right],$$

where N is a centered reduced Gaussian random variable, independent of $(\tilde{W}_s;\ s\leqslant t)$. Conditioning with respect to N, we are now facing the computation of

$$E^{\mu}\left[\exp\left(\int_0^t \tilde{W}_s v(\mathrm{d}s)+c\tilde{W}_t\right)\right]=\exp\left(\frac{1}{2}E^{\mu}\left[\left(\int_0^t \tilde{W}_s v(\mathrm{d}s)+c\tilde{W}_t\right)^2\right]\right).$$

It remains to develop the right-hand side as a second-order polynomial with respect to c, and to integrate in c, relatively to the law of $(c=)\sqrt{-F(t)}N$. A little algebra which hinges in particular upon the elementary formula

$$E\left[\exp\left(\frac{a}{2}N^2+bN\right)\right]=\frac{1}{\sqrt{1-a}}\exp\left(\frac{b^2}{2(1-a)}\right), \tag{27}$$

then yields formulas (25) and (26). We use formula (27) to calculate

$$I_{\mu,v}(t)=\exp\left(\frac{1}{2}\hat{F}(t)\right)E\left[\exp\left(\frac{1}{2}(N^2(-F(t))u(t)+2N\sqrt{-F(t)}v(t)+w(t))\right)\right],$$

hence, $a=-F(t)u(t)$, $b=\sqrt{-F(t)}v(t)$ with the following values for u, v, w:

$$u(t)=\Phi(t)\Psi(t),$$

$$v(t)=\Phi(t)\int_0^t \Psi(s)v(\mathrm{d}s),$$

$$w(t)=\int_0^t \frac{1}{\Phi^2(u)}\left(\int_u^t \Phi(s)v(\mathrm{d}s)\right)^2\mathrm{d}u.$$

Hence, we obtain

$$I_{\mu,v}(t)=\sqrt{\frac{\Phi(t)}{1-a}}\exp\left(\frac{1}{2}w(t)\right)\exp\left(\frac{b^2}{2(1-a)}\right).$$

It follows from the Wronskian's relation (23) that

$$\frac{\Phi(t)}{1-a}=\frac{1}{\Psi'(t)}\quad\text{and}\quad\frac{b^2}{2(1-a)}=-\theta(t)\left(\int_0^t \Psi(s)v(\mathrm{d}s)\right)^2,$$

which finally yields (25). □

Remark 3.1. The fact that the law $\mathbb{P}^\mu$ of $(\tilde{W}_t)_{t\leqslant 1}$ is equivalent to Wiener measure $\mathbb{P}^0$ with density $\mathrm{d}\mathbb{P}^\mu/\mathrm{d}\mathbb{P}^0 = Z_1^\mu$ is a particular case of the following general, but not so well-known result. Let $\mathbb{P}^b$ denote the law on $C([0,1])$ of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t + b(t,X_t)\mathrm{d}t, \tag{28}$$

with $X_0 = 0$, where $\{b(t,x),\ t\geqslant 0,\ x\in\mathbb{R}\}$ satisfies

(i) $|b(t,x)-b(t,y)|\leqslant C|x-y|$,
(ii) $\sup_{t\leqslant 1}|b(t,0)| < \infty$.

Then $\mathbb{P}^b$ is equivalent to Wiener measure $\mathbb{P}^0$ with density $\mathrm{d}\mathbb{P}^b/\mathrm{d}\mathbb{P}^0 \equiv Z_1^b$, where

$$Z_t^b = \exp\left(\int_0^t b(s,W_s)\mathrm{d}W_s - \frac{1}{2}\int_0^t b^2(s,W_s)\mathrm{d}s\right).$$

In particular, $(Z_t^b)_{t\leqslant 1}$ is a $\mathbb{P}^0$-martingale with expectation 1. When (i) and (ii) may not be satisfied, and (28) may be explosive, McKean (1969, pp. 66–67) presents the adequate modification of this result. Note that one should not worry about the applicability of the Novikov or Kazamaki criteria to the martingale property of $(Z_t^\mu)_{t\leqslant 1}$; indeed, these criteria were developed to take care of "general" exponential local martingales, not of those arising from (local) absolute continuity relations between solutions of stochastic differential equations.

Corollary 3.1. *With the notation*:

$$\nu_W(\mathrm{d}s) = f(s)\mathrm{d}W_s - f(s)h(s)W_s\,\mathrm{d}s,$$

formula (25) *with* ν *changed into* ν_W *is the conditional expectation of* $\mathscr{E}_{\mu,\nu_W}$ *given* W.

Proof. Note that $\int_0^t \tilde{W}_s\nu_W(\mathrm{d}s)$ is approximated by

$$\int_0^t \tilde{W}_s\nu_n(\mathrm{d}s) = \sum_{\sigma_n} \tilde{W}_{t_i} f(t_i)[(W_{t_{i+1}} - W_{t_i}) - h(t_i)W_{t_i}(t_{i+1}-t_i)],$$

where $(\sigma_n)_{n\geqslant 0}$ is a sequence of subdivisions of $[0,t]$, whose mesh goes to 0 as $n\to\infty$, and this approximation holds in the following sense: the limit occurs both in probability, and $\exp(\int_0^t \tilde{W}_s\nu_n(\mathrm{d}s))$ converges in any L^p towards $\exp(\int_0^t \tilde{W}_s\nu_W(\mathrm{d}s))$. This ensures that formula (25) also holds for ν_W. □

We now go back to our main object, that is, to compute the conditional expectation $\Lambda_t^{(2)} := E[\mathscr{E}_t^{(2)}|\mathscr{F}_t^W]$. It follows from (22), (23), Itô's product rule and Fubini theorem that

$$\begin{aligned}
&\frac{1}{2}\int_0^t\left(\int_u^t \frac{\Phi(s)}{\Phi(u)}\nu_W(\mathrm{d}s)\right)^2\mathrm{d}u - \theta(t)\left(\int_0^t \Psi(s)\nu_W(\mathrm{d}s)\right)^2\\
&\quad= \int_0^t \Psi(s)\left(\int_s^t \Phi(u)\nu_W(\mathrm{d}u)\right)\nu_W(\mathrm{d}s) - \int_0^t\left(\Phi(s) - \frac{1}{\Psi'(s)}\right)\\
&\qquad\times\left(\int_0^s \Psi(u)\nu_W(\mathrm{d}u)\right)\nu_W(\mathrm{d}s) + \frac{1}{2}\int_0^t \frac{f^2(s)}{(\Psi'(s))^2}\left(\int_0^s \Psi(u)\nu_W(\mathrm{d}u)\right)^2\mathrm{d}s
\end{aligned}$$

$$= \int_0^t f(u)k_u(W_s;\ s\leqslant u)\mathrm{d}W_u - \frac{1}{2}\int_0^t f^2(u)k_u^2(W_s;\ s\leqslant u)\mathrm{d}u$$
$$- \int_0^t f(u)h(u)W_u k_u(W_s;\ s\leqslant u)\mathrm{d}u,$$

where

$$k_u(W_s;\ s\leqslant u) := \frac{1}{\Psi'(u)}\int_0^u \Psi(v)v_W(\mathrm{d}v)$$
$$= \frac{1}{\Psi'(u)}\int_0^u \Psi(v)(f(v)\mathrm{d}W_v - f(v)h(v)W_v\,\mathrm{d}v). \tag{29}$$

Therefore, we derive

$$\Lambda_t = \exp\left\{\int_0^t (f(u)k_u(W_v;\ v\leqslant u) + h(u)W_u)\mathrm{d}W_u \right.$$
$$\left. -\frac{1}{2}\int_0^t (f(u)k_u(W_v;\ v\leqslant u) + h(u)W_u)^2\mathrm{d}u\right\}. \tag{30}$$

From (21), the definition of Λ_t and Girsanov's transformation, we can get the main result in this section, i.e., the canonical decomposition of X as a semimartingale in its own filtration $(\mathscr{F}_t^X)$.

Theorem 3.1. *The canonical decomposition of* $(X_t)_{0\leqslant t<1}$ *is given by*

$$X_t = B_t + \int_0^t (f(u)k_u(X_v;\ v\leqslant u) + h(u)X_u)\mathrm{d}u, \tag{31}$$

where $(B_t)_{0\leqslant t<1}$ *is a standard Brownian motion with respect to* $(\mathscr{F}_t^X)_{0\leqslant t<1}$, *and the functional* $(k_t(X_v;\ v\leqslant t))_{0\leqslant t<1}$ *is of the form* (29).

Remark 3.2. (1) Comparing (20) and (31) and recalling the first assertion of Lemma 2.3, we see that

$$E[\tilde{W}_t|\mathscr{F}_t^X] = k_t(X_s;\ s\leqslant t). \tag{32}$$

In particular, we have obtained the explicit form of the projection appearing in Remark 2.1.

(2) We have obtained that the identification of the conditional expectation in (32) holds for $f, h \in \mathscr{A}(0,1)$. In fact, it is enough to assume that f and h belong to the set

$$\bar{\mathscr{A}}(0,1) := \left\{\varphi : f \text{ is measurable}, \int_0^t \sqrt{s}|\varphi(s)|\,\mathrm{d}s < \infty \text{ for all } 0\leqslant t<1\right\}.$$

Note that $\mathscr{A}(0,1)\subset\bar{\mathscr{A}}(0,1)$, since

$$\left(\int_0^t \sqrt{s}|\varphi(s)|\mathrm{d}s\right)^2 \leqslant t\int_0^t s\varphi^2(s)\mathrm{d}s$$

by the Cauchy–Schwarz inequality. It is not difficult to show that for all $s\leqslant t$,

$$E\left[X_s\left(\tilde{W}_t - \frac{1}{\Psi'(t)}\int_0^t \Psi(u)(f(u)\mathrm{d}X_u - f(u)h(u)X_u\,\mathrm{d}u)\right)\right] = 0,$$

where $\Psi(t)$ satisfies Sturm–Liouville equation (24). This implies (32).

3.2. Volterra representation and canonical decomposition

Hitsuda (1968) shows that the law of a Gaussian process $(X_t)_{0\leqslant t\leqslant 1}$ with $E(X_t)=0$ is equivalent to Wiener measure if and only if X_t can be represented in the form

$$X_t = B_t + \int_0^t \int_0^s l(s,u)\mathrm{d}B_u\,\mathrm{d}s, \tag{33}$$

where B is a Brownian motion and $l(s,u)$ is a square-integrable Volterra kernel, i.e., a measurable function on $[0,1]\times[0,1]$ such that

(i) $\int_0^1 \int_0^s l^2(s,u)\mathrm{d}u\,\mathrm{d}s < \infty$.
(ii) $l(s,u)=0$ for $0<s<u<1$.

This representation is unique in the sense that if X has another representation

$$X_t = \tilde{B}_t + \int_0^t \int_0^s \tilde{l}(s,u)\mathrm{d}\tilde{B}_u\,\mathrm{d}s,$$

then $B=\tilde{B}$ and $l(s,u)=\tilde{l}(s,u)$ for almost all $s,u\in[0,1]$; see Hida and Hitsuda (1993). We shall call the representation (33) the *Volterra representation* of X.

Proposition 3.2. *X and B generate the same filtration. Consequently, the Volterra representation* (33) *is the canonical decomposition of X as a semimartingale in its own filtration* $(\mathscr{F}_t^X)$.

Proof. Given a square-integrable Volterra kernel l, there is a unique square-integrable Volterra kernel R_l which satisfies the equations

$$\begin{aligned} &l(t,s)+R_l(t,s)+\int_s^t l(t,u)R_l(u,s)\mathrm{d}u=0,\\ &l(t,s)+R_l(t,s)+\int_s^t R_l(t,u)l(u,s)\mathrm{d}u=0; \end{aligned} \tag{34}$$

for almost all $s<t$. We call R_l the resolvent kernel of l; see Yosida (1991, Chapter 4) or Hida and Hitsuda (1993). As in Hida and Hitsuda (1993, pp. 136–137), we can now use the kernel R_l in order to reconstruct B in terms of X:

$$\begin{aligned} &\mathrm{d}X_t + \int_0^t R_l(t,u)\mathrm{d}X_u\,\mathrm{d}t\\ &\quad = \mathrm{d}B_t + \int_0^t l(t,u)\,\mathrm{d}B_u\,\mathrm{d}t + \left(\int_0^t R_l(t,u)\left(\mathrm{d}B_u + \int_0^u l(u,v)\mathrm{d}B_v\,\mathrm{d}u\right)\right)\mathrm{d}t\\ &\quad = \mathrm{d}B_t + \int_0^t \left(l(t,u)+R_l(t,u)+\int_u^t R_l(t,v)l(v,u)\mathrm{d}v\right)\mathrm{d}B_u\,\mathrm{d}t\\ &\quad = \mathrm{d}B_t, \end{aligned}$$

i.e., we have

$$X_t = B_t + \int_0^t \int_0^s l(s,u)\mathrm{d}B_u\,\mathrm{d}s, \tag{35}$$

and

$$B_t = X_t + \int_0^t \int_0^s R_l(s,u)\mathrm{d}X_u\,\mathrm{d}s. \tag{36}$$

Thus, X and B generate the same filtration. Hence (33) is the canonical decomposition of X in its own filtration. □

Remark 3.3. Let us make explicit the square-integrable Volterra kernel corresponding to our situation (20) in the special case where $h(t)=0$, i.e.,

$$X_t = W_t + \int_0^t f(u)\tilde{W}_u\,\mathrm{d}u.$$

We have shown that the canonical decomposition of X is given by

$$X_t = B_t + \int_0^t \frac{f(s)}{\Psi'(s)} \int_0^s f(u)\Psi(u)\mathrm{d}X_u\,\mathrm{d}s. \tag{37}$$

Integrating $f(t)\Psi(t)$ on both sides and using Itô's product rule, we obtain

$$\int_0^t f(u)\Psi(u)\mathrm{d}X_u = \Psi'(t)\int_0^t \frac{f(v)\Psi(v)}{\Psi'(v)}\mathrm{d}B_v.$$

Thus, the representation (37) takes the form

$$X_t = B_t + \int_0^t f(s)\int_0^s \frac{f(u)\Psi(u)}{\Psi'(u)}\mathrm{d}B_u\,\mathrm{d}s. \tag{38}$$

This shows that the corresponding square-integrable Volterra kernel is given by

$$l(s,u) = \frac{f(s)f(u)\Psi(u)}{\Psi'(u)},$$

and we read from (37) that

$$R_l(s,u) = -\frac{f(s)}{\Psi'(s)}f(u)\Psi(u). \tag{39}$$

One can verify from (39) that R_l satisfies indeed the two characteristic equations in (34).

Remark 3.4. The study of the general equation (20) for the pair (f,h) can be reduced to that of $(f,0)$. Suppose a process (X_t) satisfies (20) and introduce the Gaussian process

$$\xi_t = W_t + \int_0^t f(u)\tilde{W}_u\,\mathrm{d}u.$$

Remark that (X_t) and (ξ_t) have the same filtration, since

$$\xi_t = X_t - \int_0^t h(u)X_u\,\mathrm{d}u, \tag{40}$$

and

$$X_t = \int_0^t \exp\left(\int_s^t h(u)\mathrm{d}u \right) \mathrm{d}\xi_s.$$

Using the canonical decomposition for (ξ_t) given by (38), we obtain the canonical decomposition

$$X_t = B_t + \int_0^t \left(f(s) \int_0^s \frac{f(u)\Psi(u)}{\Psi'(u)} \mathrm{d}B_u + h(s)X_s \right) \mathrm{d}s \tag{41}$$

for X. The pathwise solution of this equation leads us to the alternative representation

$$X_t = \int_0^t \exp\left(\int_s^t h(u)\,\mathrm{d}u \right) \mathrm{d}B_s + \int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)}$$
$$\times \left(\int_u^t f(v) \exp\left(\int_v^t h(w)\,\mathrm{d}w \right) \mathrm{d}v \right) \mathrm{d}B_u \tag{42}$$

as a functional of the Brownian motion B, in analogy to (38). From Allinger and Mitter (1981) and Davis (1977) we know that the σ-algebra $\mathscr{F}_t^X$ coincides with $\mathscr{F}_t^B$ for all t, up to some $\mathbb{P}$ null sets. Thus, $\mathscr{F}_t^\xi = \mathscr{F}_t^B = \mathscr{F}_t^X$. In the terminology of Hida and Hitsuda (1993), representation (42) is called "*the canonical representation of X relative to B*", and B is the innovation process for X.

3.3. A class of path-dependent transformations

Using the same method as above, we can extend our result in Theorem 3.1 as follows. Consider a process $(X_t)_{0\leqslant t<1}$ satisfying the stochastic functional differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t + \left(f(t)\tilde{W}_t + \int_0^t h(t,s)\mathrm{d}X_s \right) \mathrm{d}t, \tag{43}$$

for some deterministic function $f \in C^1(0,1) \cap \mathscr{A}(0,1)$, a function $h \in C^{1,1}((0,1) \times (0,1))$ with $\int_0^t \int_0^s h^2(s,u)\mathrm{d}u\,\mathrm{d}s < \infty$ for all $t < 1$.

Theorem 3.2. *The canonical decomposition of* $(X_t)_{0\leqslant t<1}$ *is*

$$\mathrm{d}X_t = \mathrm{d}B_t + \left(f(t)k_t(X_s;\ s\leqslant t) + \int_0^t h(t,s)\mathrm{d}X_s \right) \mathrm{d}t,$$

where $(B_t)_{0\leqslant t\leqslant 1}$ *is an* $(\mathscr{F}_t^X)$*-Brownian motion, and* $k_u(X_s; s\leqslant u)$ *is of the form*

$$\frac{1}{\Psi'(u)} \int_0^u \Psi(v)f(v) \left(\mathrm{d}X_v - \left(\int_0^v h(v,r)\mathrm{d}X_r \right) \mathrm{d}v \right). \tag{44}$$

Proof. We set

$$\nu_W(\mathrm{d}s) := f(s)\mathrm{d}W_s - f(s) \int_0^s h(s,u)\mathrm{d}W_u\,\mathrm{d}s.$$

The rest of the proof is similar to the proof in Section 3.1. □

3.4. Some examples

We look at some processes whose canonical decompositions take a simple form.

Example 3.1. Consider a process $(X_t)_{0\leqslant t<1}$ satisfying the stochastic differential equation:

$$\mathrm{d}X_t = \mathrm{d}W_t + \frac{a}{1-t}(\tilde{W}_t - X_t)\mathrm{d}t,$$

that is, $f(t) = -h(t) = a/(1-t)$, with a nonzero constant a. Then the corresponding Sturm–Liouville equation is

$$\Phi''(u) = \frac{a^2}{(1-u)^2}\Phi(u), \tag{45}$$

which, arguably perhaps, is one of the simplest cases where the Sturm–Liouville equation has elementary solutions. Indeed, it is immediate to check that a function $(1-u)^\lambda$ solves (45) if and only if $\lambda(\lambda-1) = a^2$, an equation which admits the two solutions: $\lambda_+(a)$ and $\lambda_-(a)$ given by

$$\lambda_\pm(a) := \tfrac{1}{2} \pm \sqrt{a^2 + \tfrac{1}{4}}.$$

Clearly, $\lambda_-(a) < 0 < \lambda_+(a)$. Thus, the decreasing solution of (45) is

$$\Phi(u) = (1-u)^{\lambda_+(a)}.$$

And from the definition and the boundary condition of $\Psi(u)$ we can get

$$\Psi(u) = \frac{(1-u)^{\lambda_-(a)} - (1-u)^{\lambda_+(a)}}{\sqrt{1+4a^2}}.$$

From (29), the conditional expectation of $\tilde{W}_t$ relative to $\mathscr{F}_t^X$ is given by

$$\begin{aligned} &k_u(X_s;\ s\leqslant u) \\ &:= \frac{1}{\Psi'(u)}\int_0^u \Psi(v)\left(\frac{a}{1-v}\mathrm{d}X_v + \frac{a^2}{(1-v)^2}X_v\,\mathrm{d}v\right) \\ &= X_u + a\int_0^u \frac{(\lambda_-(a)+1)(1-v)^{-\lambda_+(a)} - (\lambda_+(a)+1)(1-v)^{-\lambda_-(a)}}{\lambda_+(a)(1-u)^{-\lambda_-(a)} - \lambda_-(a)(1-u)^{-\lambda_+(a)}}\mathrm{d}X_v. \end{aligned}$$

In particular, if $a = 1$, then $\lambda_+(1) = A$ and $\lambda_-(1) = B$, defined as in Lemma 2.4, and we are led to the same result as in Section 2.

Example 3.2. Consider the simple example:

$$\mathrm{d}X_t = \mathrm{d}W_t + a(\tilde{W}_t - X_t)\mathrm{d}t,$$

with a nonzero constant a. The desired solution of the corresponding Sturm–Liouville equation is

$$\Psi(t) = \frac{1}{2a}(\mathrm{e}^{at} - \mathrm{e}^{-at}),$$

and the conditional expectation of $\tilde{W}_t$ with respect to $\mathscr{F}_t^X$ is given by

$$k_t(X_s;\ s\leqslant t)=X_t-\frac{2}{\mathrm{e}^{at}+\mathrm{e}^{-at}}\int_0^t \mathrm{e}^{-au}\,\mathrm{d}X_u.$$

Therefore, the canonical decomposition of X_t has the form

$$X_t=B_t-\int_0^t \frac{2a}{\mathrm{e}^{au}+\mathrm{e}^{-au}}\left(\int_0^u \mathrm{e}^{-av}\,\mathrm{d}X_v\right)\mathrm{d}u,$$

where $(B_t)_{0\leqslant t\leqslant 1}$ is a Brownian motion relative to $(\mathscr{F}_t^X)_{0\leqslant t\leqslant 1}$.

Example 3.3. Let $(W_t)_{t\geqslant 0}$ and $(\tilde{W}_t)_{t\geqslant 0}$ be two independent Brownian motions, starting from 0. The process $(X_t)_{t\geqslant 0}$ satisfies the stochastic differential equation (20) with $f(t)=-k/t$ with a constant k and $h\equiv 0$. In Section 3.2 we have already discussed a few results about the case with $h\equiv 0$. First, we have to solve

$$\Psi''(s)=\frac{k^2}{s^2}\Psi(s),$$

and then single out a solution with $\Psi(0)=0$. It is easy to check that $\Psi(s)=s^{\lambda}$ with $\lambda=\frac{1}{2}+\sqrt{k^2+\frac{1}{4}}$ is the wanted solution. Now formula (38) gives

$$X_t=B_t+\frac{k^2}{\lambda}\int_0^t \frac{B_u}{u}\mathrm{d}u, \tag{46}$$

with a Brownian motion (B_t). In Jeulin and Yor (1990, Theorem 9) it has been shown that

$$X_t^{\mu}:=B_t-\mu\int_0^t \frac{B_s}{s}\mathrm{d}s \tag{47}$$

has a strictly smaller filtration than the filtration of B iff $\mu>\frac{1}{2}$. Coming back to formula (46), and comparing with (47), we find

$$\mu=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1+4k^2}.$$

Thus, in our study, μ is always negative, hence $(X_t)_{t\geqslant 0}$ has the same filtration as (B_t).

4. Application of stochastic filtering theory for Gaussian processes

In Section 3 we have computed the canonical representation of our transformations (20) of two Brownian motions by direct methods. As an alternative, we can derive them as corollaries of the stochastic filtering theory for Gaussian processes. At the same time, this allows us to extend our results to a general class of transformations where the drift Y in (1) is given as an adapted linear functional of X and $\tilde{W}$.

Suppose that the process X satisfies a stochastic differential equation of the form

$$\mathrm{d}X_t=\mathrm{d}W_t+\left(\int_0^t F(t,u)\mathrm{d}\tilde{W}_u+\int_0^t H(t,u)\mathrm{d}X_u\right)\mathrm{d}t, \tag{48}$$

with $X_0 = 0$, where $(\tilde{W}_t)_{0\leqslant t\leqslant 1}, (W_t)_{0\leqslant t\leqslant 1}$ are two independent Brownian motions, and F, H are square-integrable Volterra kernels on $(0,1)\times(0,1)$, i.e., they satisfy the conditions in Section 3.2. Note that the processes considered in (20) belong to this class.

Lemma 4.1. *There is a unique Brownian motion B and a unique square-integrable Volterra kernel G_F such that*

$$W_t + \int_0^t \int_0^s F(s,u)\mathrm{d}\tilde{W}_u \,\mathrm{d}s = B_t + \int_0^t \int_0^s G_F(s,u)\mathrm{d}B_u \,\mathrm{d}s.$$

The kernel G_F is determined by the equation

$$\int_0^s F(t,v)F(s,v)\mathrm{d}v = G_F(t,s) + \int_0^s G_F(t,v)G_F(s,v)\mathrm{d}v. \tag{49}$$

Moreover the natural filtration of B is identical to that of the left-hand side.

Proof. (1) Let us denote by Z the "signal process"

$$Z_t = \int_0^t F(t,u)\mathrm{d}\tilde{W}_u,$$

and by ξ the "observation process"

$$\xi_t = W_t + \int_0^t Z_s \,\mathrm{d}s = W_t + \int_0^t \int_0^s F(s,u)\mathrm{d}\tilde{W}_u \,\mathrm{d}s. \tag{50}$$

From Lemma 2.3 we know that ξ_t can be written as

$$\xi_t = B_t + \int_0^t E[Z_s|\mathscr{F}_s^\xi]\mathrm{d}s,$$

where (B_t) is an $(\mathscr{F}_t^\xi)$-Brownian motion. Since $E[Z_t|\mathscr{F}_t^\xi](\omega)$ can be chosen (t,ω)-measurable and $(\mathscr{F}_t^\xi)$-adapted, we can write

$$E[Z_t|\mathscr{F}_t^\xi] = \gamma(t,\xi(w)),$$

where γ is a non-anticipative functional in the sense of Kallianpur (1980, Definition 5.1.1). Furthermore, it follows from

$$E\int_0^t (E[Z_u|\mathscr{F}_u^\xi])^2\mathrm{d}u \leqslant E\int_0^t Z_u^2 \,\mathrm{d}u = \int_0^t \int_0^s (F(s,u))^2 \,\mathrm{d}u \,\mathrm{d}s < \infty$$

that

$$\int_0^t \gamma^2(s,\xi(\omega))\mathrm{d}s < \infty, \qquad \mathbb{P}\text{-a.s.},$$

for all $t < 1$. By Kallianpur (1980, Theorem 9.4.1) and Section 3.2, there is a unique square-integrable Volterra kernel G_F such that the Gaussian process ξ has the representation

$$\xi_t = B_t + \int_0^t \int_0^s G_F(s,u)\mathrm{d}B_u \,\mathrm{d}s. \tag{51}$$

Moreover we have $(\mathscr{F}_t^\xi) = (\mathscr{F}_t^B)$, due to Proposition 3.2.

(2) Since W and $\tilde{W}$ are two independent Brownian motions, we get from (50), for $s \leqslant t$,

$$E[\xi_s \xi_t] = E[W_s W_t] + E\left[\int_0^s \int_0^u F(u,r) \mathrm{d}\tilde{W}_r \,\mathrm{d}u \int_0^t \int_0^v F(v,q) \mathrm{d}\tilde{W}_q \,\mathrm{d}v\right]$$

$$= s + 2\int_0^s \int_0^u \int_0^v F(u,r)F(v,r) \mathrm{d}r \,\mathrm{d}v \,\mathrm{d}u$$

$$+ \int_s^t \int_0^s \int_0^v F(u,r)F(v,r) \mathrm{d}r \,\mathrm{d}v \,\mathrm{d}u. \tag{52}$$

We can also compute the covariance of ξ from (51):

$$E[\xi_s \xi_t] = E\left[\left(B_s + \int_0^s \int_0^u G_F(u,r) \mathrm{d}B_r \,\mathrm{d}u\right)\left(B_t + \int_0^t \int_0^v G_F(v,q) \mathrm{d}B_q \,\mathrm{d}v\right)\right]$$

$$= s + 2\int_0^s \int_0^u G_F(u,v) \mathrm{d}v \,\mathrm{d}u + \int_s^t \int_0^s G_F(u,v) \mathrm{d}v \,\mathrm{d}u$$

$$+2\int_0^s \int_0^u \int_0^v G_F(u,r)G_F(v,r) \mathrm{d}r \,\mathrm{d}v \,\mathrm{d}u$$

$$+ \int_s^t \int_0^s \int_0^v G_F(u,r)G_F(v,r) \mathrm{d}r \,\mathrm{d}v \,\mathrm{d}u. \tag{53}$$

Since the right-hand sides of these two equations much coincide, differentiating first with respect to t then with respect to s yields (49) for almost every $s \leqslant t$. □

Remark 4.1. In the notation of Kallianpur (1980, p. 235), Eq. (49) can be viewed as the factorization $S = (I + G)(I + G^*)$ of the integral operator S defined by $I + FF^*$, where F, G are integral operators with square-integrable Volterra kernels $F(t,s)$ and $G_F(t,s)$, respectively, i.e., for all $f, g \in L^2(0,1)$,

$$\langle (I + FF^*)f, g\rangle = \langle (I + G)(I + G^*)f, g\rangle.$$

In order to see this, let $f(u) = I_{(0,s)}(u)$ and $g(u) = I_{(0,t)}(u)$ with $0 \leqslant s \leqslant t \leqslant 1$. Using the properties of Volterra kernels, we have

$$\langle (I + FF^*)f, g\rangle = \langle f, g\rangle + \langle F^* f, F^* g\rangle$$

$$= \int_0^1 f(u)g(u) \,\mathrm{d}u + \int_0^1 \left(\int_0^1 F(v,u)f(v) \mathrm{d}v\right)$$

$$\times \left(\int_0^1 F(r,u)g(r) \mathrm{d}r\right) \mathrm{d}u$$

$$= s + \int_0^s \left(\int_u^s F(v,u) \mathrm{d}v\right)\left(\int_u^t F(r,u) \mathrm{d}r\right) \mathrm{d}u,$$

which equals to the right-hand side of (52). On the other hand,

$$\begin{aligned}\langle (I+G)(I+G^*)f,g\rangle &= \langle (I+G^*)f,(I+G^*)g\rangle \\ &= \int_0^1 \left(f(u) + \int_0^1 G_F(v,u)f(v)\mathrm{d}v \right) \\ &\quad \times \left(g(u) + \int_0^1 G_F(v,u)g(v)\mathrm{d}v \right) \mathrm{d}u \\ &= \int_0^s \left(1 + \int_u^s G_F(v,u)\mathrm{d}v \right) \left(1 + \int_u^t G_F(v,u)\mathrm{d}v \right) \mathrm{d}u,\end{aligned}$$

which is exactly the right-hand side of (53).

Now, we look at the canonical decomposition of the process X given by (48).

Theorem 4.1. *The canonical decomposition of X as a semimartingale in its own filtration $(\mathscr{F}_t^X)$ is given by*

$$\mathrm{d}X_t = \mathrm{d}B_t + \left(\int_0^t G_F(t,u)\mathrm{d}B_u + \int_0^t H(t,u)\mathrm{d}X_u \right) \mathrm{d}t. \tag{54}$$

Moreover we have $(\mathscr{F}_t^X) = (\mathscr{F}_t^B)$.

Proof. From (50) and (54), we have

$$\mathrm{d}X_t = \mathrm{d}\xi_t + \int_0^t H(t,u)\mathrm{d}X_u \,\mathrm{d}t. \tag{55}$$

As in (34), let R_{-H} denote the resolvent kernel of the square-integrable Volterra kernel $-H$. Due to Eqs. (35) and (36), we have

$$\xi_t = X_t - \int_0^t \int_0^s H(s,u)\mathrm{d}X_u \,\mathrm{d}s,$$

and

$$X_t = \xi_t + \int_0^t \int_0^s R_{-H}(s,u)\mathrm{d}\xi_u \,\mathrm{d}s.$$

These two equations together with Lemma 4.1 imply $\mathscr{F}_t^X = \mathscr{F}_t^\xi = \mathscr{F}_t^B$, for all t. Hence, (B_t) is also an $(\mathscr{F}_t^X)$-Brownian motion. Substituting representation (51) in Eq. (55), we obtain (54). $\square$

Remark 4.2. Comparing (48), (54) and Lemma 2.3, we see that

$$E\left[\int_0^t F(t,u)\mathrm{d}\tilde{W}_u | \mathscr{F}_t^X \right] = \int_0^t G_F(t,u)\mathrm{d}B_u.$$

Let us now consider the special case when F admits a factorization $F(t,s) = f(t)g(s)$ for some functions $f,g \in C^1(0,1)$ which satisfy

$$\int_0^t \int_0^u f^2(u)g^2(v)\mathrm{d}v \,\mathrm{d}u < \infty, \tag{56}$$

for all $t < 1$.

Corollary 4.1. *Suppose the process* $(X_t)_{0\leqslant t<1}$ *satisfies*

$$\mathrm{d}X_t=\mathrm{d}W_t+\left(f(t)\int_0^t g(u)\mathrm{d}\tilde{W}_u+\int_0^t H(t,u)\mathrm{d}X_u\right)\mathrm{d}t, \tag{57}$$

with $f,g\in C^1(0,1)$ *satisfying* (56), $f\neq 0$ *a.s., and a square-integrable Volterra kernel* $H(t,s)$. *Then the canonical decomposition of* X *is of the form*

$$\mathrm{d}X_t=\mathrm{d}B_t+\left(f(t)\int_0^t \alpha(u)\mathrm{d}B_u+\int_0^t H(t,u)\mathrm{d}X_u\right)\mathrm{d}t, \tag{58}$$

where the function $\alpha(t)$ *is the solution of the differential equation*

$$\left(\frac{\alpha(t)}{f(t)}\right)'+\alpha^2(t)=g^2(t), \tag{59}$$

with boundary condition $\alpha(0)=0$.

Proof. We have only to prove that $G_F(t,s)=f(t)\alpha(s)$, where α satisfies (59) and $\alpha(0)=0$, is the solution of (49). In fact, the right-hand side of (49) is equal to

$$\begin{aligned}
&G_F(t,s)+\int_0^s G_F(t,u)G_F(s,u)\mathrm{d}u\\
&\quad=f(t)\alpha(s)+f(t)f(s)\int_0^s\alpha^2(u)\mathrm{d}u\\
&\quad=f(t)\alpha(s)+f(t)f(s)\left[\int_0^s g^2(u)\mathrm{d}u-\frac{\alpha(s)}{f(s)}\right]\\
&\quad=f(t)f(s)\int_0^s g^2(u)\mathrm{d}u=\int_0^s F(t,u)F(s,u)\mathrm{d}u,
\end{aligned}$$

which is exactly the left-hand side of (49). □

In order to see the connection with our discussion in the preceding sections, let us set $g\equiv 1$ and $H(t,u)=h(t)$. Then Eq. (59) can be written as

$$\left(\frac{\alpha(t)}{f(t)}\right)'+\alpha^2(t)=1,$$

with $\alpha(0)=0$. The corresponding solution is given by

$$\alpha(t)=\frac{f(t)\Psi(t)}{\Psi'(t)}, \tag{60}$$

where $\Psi(t)$ is the solution of the Sturm–Liouville equation (24). Substituting this result in (58), we see that the result coincides with (41).

Remark 4.3. The discussion of Eq. (57) can be reduced to the case $g\equiv 1$. All we need is to consider $H=0$. Then, we have

$$g(t)\mathrm{d}X_t=g(t)\mathrm{d}W_t+f(t)g(t)\int_0^t g(u)\mathrm{d}\tilde{W}_u\,\mathrm{d}t.$$

Let us introduce the (time-changed) processes $(\hat{X}_u)$, $(\hat{W}_u)$ and $(\hat{\tilde{W}}_u)$ defined as:

$$\int_0^t g(v)\mathrm{d}X_v = \hat{X}_{G(t)},$$

where

$$G(t) = \int_0^t g^2(v)\mathrm{d}v,$$

etc. We obtain

$$\mathrm{d}\hat{X}_u = \mathrm{d}\hat{W}_u + \varphi(u)\hat{\tilde{W}}_u\,\mathrm{d}u,$$

where

$$\varphi(u) = \left(\frac{f}{g}\right)(G^{-1}(u)).$$

Consequently, we obtain the canonical decomposition of $\hat{X}$:

$$\hat{X}_t = \hat{B}_t + \varphi(t)\int_0^t \hat{\alpha}(u)\,\mathrm{d}\hat{B}_u,$$

where

$$\hat{\alpha}(t) = \varphi(t)\hat{\psi}(t)/\hat{\psi}'(t),$$

and $\hat{\psi}$ is our usual notation for the solution of $u'' = \varphi^2 u$. It now remains to undo the time-change, and relate $\hat{\alpha}$ to α, as given in (59).

5. How to get again a Brownian motion

In our initial example (10), where the final value $\tilde{W}_1$ was known in advance, the solution $(X_t)_{0\leqslant t\leqslant 1}$ was again a Brownian motion, and it was tied to the final value $\tilde{W}_1$. It is also possible to satisfy both conditions in our modified situation when at time t we only know the past of $\tilde{W}$?

5.1. Characterization of Brownian motions

Consider a process $(X_t)_{0\leqslant t<1}$ with $X_0 = 0$ which satisfies the stochastic functional differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t + \left(\int_0^t F(t,u)\mathrm{d}\tilde{W}_u + \int_0^t H(t,u)\mathrm{d}X_u\right)\mathrm{d}t, \tag{61}$$

with square-integrable Volterra kernels F and H.

Theorem 5.1. *A process X satisfying* (61) *is a Wiener process with respect to its own filtration* $(\mathscr{F}_t^X)$ *if and only if* $H(t,s) = -G_F(t,s)$, *where* G_F *is the square-integrable Volterra kernel defined by* (49). *In other words, X is of the form*

$$X_t = W_t + \int_0^t \left(\int_0^s F(s,u)\mathrm{d}\tilde{W}_u - \int_0^s G_F(s,u)\mathrm{d}X_u\right)\mathrm{d}s. \tag{62}$$

Proof. (1) Suppose X is a Wiener process with respect to its own filtration $(\mathscr{F}_t^X)$. By uniqueness of the Doob–Meyer decomposition in $(\mathscr{F}_t^X)$, our representation (54) implies $B = X$ and

$$\int_0^t [G_F(t,u) + H(t,u)]\mathrm{d}X_u = 0, \tag{63}$$

$\mathbb{P}$-a.s. for all t. But (63) implies

$$G_F(t,u) + H(t,u) = 0,$$

for almost all $u \leqslant t$, since X is a Brownian motion.

(2) Conversely, assume that X has the form (62). The canonical representation (54) implies

$$X_t = B_t + \int_0^t \left(\int_0^s G_F(s,u)\mathrm{d}B_u - \int_0^s G_F(s,u)\mathrm{d}X_u \right) \mathrm{d}s,$$

i.e.,

$$X_t + \int_0^t \int_0^s G_F(s,u)\mathrm{d}X_u \,\mathrm{d}s = B_t + \int_0^t \int_0^s G_F(s,u)\mathrm{d}B_u \,\mathrm{d}s.$$

We can now apply the reconstruction argument in Section 3.2 in order to conclude $X = B$. In other words, X is a Brownian motion. □

Remark 5.1. The last argument shall be taken up again in Lemma 5.1.

Let us look at the special case considered in Corollary 4.1, where F is of the form $F(t,s) = f(t)g(s)$ for some continuously differentiable functions f and g satisfying (56).

Corollary 5.1. *A process $(X_t)_{0 \leqslant t < 1}$ satisfying* (57) *is a Brownian motion if and only if*

$$H(t,u) = -f(t)\alpha(u),$$

where $\alpha(t)$ is the solution of (59) *with boundary condition $\alpha(0) = 0$. In other words, if $(X_t)_{0 \leqslant t < 1}$ is a Brownian motion with respect to its own filtration, it must be of the form*

$$\mathrm{d}X_t = \mathrm{d}W_t + f(t) \left(\int_0^t g(u)\mathrm{d}\tilde{W}_u - \int_0^t \alpha(u)\mathrm{d}X_u \right) \mathrm{d}t. \tag{64}$$

In particular, if $g \equiv 1$, then (64) *can be written as*

$$\mathrm{d}X_t = \mathrm{d}W_t + f(t) \left(\tilde{W}_t - \int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)} \mathrm{d}X_u \right) \mathrm{d}t, \tag{65}$$

where $\Psi(t)$ is the solution of the Sturm–Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t), \tag{66}$$

with $\Psi(0) = 0$ and $\Psi'(0+) = 1$.

Proof. In order to get the characterization for this class of Brownian motions from Theorem 5.1, we have only to compute $G_F(t,u)$. From the proof in Corollary 4.1, we know that $G_F(t,u)=f(t)\alpha(u)$. Substituting this result in Theorem 5.1, we get the first assertion. As to the case $g\equiv 1$, simply substitute (60) in (64). □

Corollary 5.2. *Let X be a process satisfying the stochastic differential equation* (20) *with $f,h\in C^1(0,1)\cap\mathscr{A}(0,1)$. If one of the functions $f(t)$ and $h(t)$ is not equal to* 0, *then X cannot be a Brownian motion.*

Proof. If X satisfying (20) is a Brownian motion, then it follows from Corollary 5.1 that f must be of the form

$$f(t)\Psi(t)=c\Psi'(t), \tag{67}$$

for some non-zero constant c. Substituting (67) in (66), we get

$$\Psi''(t)=cf(t)\Psi'(t).$$

Hence, the corresponding solution of the Sturm–Liouville equation is given by

$$\Psi(t)=\int_0^t \exp\left(c\int_0^u f(v)\mathrm{d}v\right)\mathrm{d}u.$$

Substituting this solution again in (67), and taking derivatives on both sides with respect to t, we have

$$cf'(t)+(1-c^2)f^2(t)=0.$$

This implies

$$f(t)=\frac{c}{1-c^2}\frac{1}{t},$$

which does not belong to $\mathscr{A}(0,1)$. □

In the class of adapted linear drift of the form (61), Theorem 5.1 characterizes those cases where the resulting process X in (62) is a new Brownian motion. Let us now return to the question whether such a Brownian motion can be tied to the endpoint $\tilde{W}_1$ of the Brownian motion $\tilde{W}$. This turns out to be impossible as long as we insist on an adapted drift, even if we drop linearity.

Proposition 5.1. *Consider any drift $(Y_t)_{0\leqslant t\leqslant 1}$ adapted to $(\mathscr{F}_t)$ such that*

$$X_t=W_t+\int_0^t Y_s\,\mathrm{d}s,$$

is a Brownian motion. If Z is any $(\mathscr{F}_t)$-Brownian motion such that $X_1=Z_1,\mathbb{P}$-a.s., then we have $Z_t=X_t=W_t$, and in particular, $Y_t=0$, $\mathrm{d}t\times\mathrm{d}\mathbb{P}$-a.s.

Proof. If $X_1=Z_1$, then

$$X_t=E[X_1|\mathscr{F}_t^X]=E[Z_1|\mathscr{F}_t^X]=E[Z_t|\mathscr{F}_t^X].$$

This implies

$$E[X_tZ_t]=E[X_t^2]=t,$$

hence

$$E[(X_t - Z_t)^2] = E[X_t^2] + E[Z_t^2] - 2E[X_t Z_t] = 0.$$

Hence, $Z_t = X_t$ is an $(\mathscr{F}_t)$-Brownian motion, and so: $X_t = W_t$. □

In our situation the Brownian motion $Z = \tilde{W}$ is independent of W, and so the proposition shows that it is impossible to obtain $X_1 = \tilde{W}_1$.

Remark 5.2. (i) Consider a square-integrable $(\mathscr{F}_t)$-adapted process $(X_t)_{t\leqslant 1}$ which is a martingale in its own filtration $(\mathscr{F}_t^X)$. Let (M_t) be a square-integrable $(\mathscr{F}_t)$-martingale such that $X_1 = M_1$ and $E[X_t^2] = E[M_t^2]$ for every $t\leqslant 1$. The proof of Proposition 5.1 shows that this implies $X_t = M_t$ every $t\leqslant 1$.

(ii) As a special case of (i), assume that (X_t) is a Brownian motion in its own filtration $(\mathscr{F}_t^X)$. Then (X_t) is an $(\mathscr{F}_t)$-Brownian motion if and only if $E[X_1|\mathscr{F}_t]$ is a Brownian motion.

(iii) As an example of (ii), consider a Brownian motion (B_t) with respect to $(\mathscr{F}_t)$. We know that

$$X_t := B_t - \int_0^t \frac{B_s}{s}\,\mathrm{d}s$$

defines a new Brownian motion $(X_t)_{t\leqslant 1}$ which is not a Brownian motion with respect to $(\mathscr{F}_t)$; cf., e.g., Yor (1992). It follows from (ii) that $(E[X_1|\mathscr{F}_t])_{t\leqslant 1}$ cannot be a Brownian motion. In fact, a direct computation shows that

$$\begin{aligned}E[X_1|\mathscr{F}_t] &= B_t - \int_0^t \frac{B_s}{s}\,\mathrm{d}s - \int_t^1 \frac{E[B_s|\mathscr{F}_t]}{s}\,\mathrm{d}s\\ &= B_t(1+\log t) - \int_0^t \frac{B_s}{s}\,\mathrm{d}s = \int_0^t (1+\log s)\,\mathrm{d}B_s.\end{aligned}$$

5.2. Enlargement of a Brownian filtration

Consider a Brownian motion X arising as the solution of the linear functional stochastic differential equation (62). Let us represent X directly in terms of the two Wiener processes W and $\tilde{W}$. To this end, we introduce the resolvent kernel L_F of G_F, i.e., G_F and L_F satisfy the following relations:

$$\begin{aligned}&G_F(t,s) + L_F(t,s) + \int_s^t L_F(t,u)G_F(u,s)\,\mathrm{d}u = 0,\\ &G_F(t,s) + L_F(t,s) + \int_s^t G_F(t,u)L_F(u,s)\,\mathrm{d}u = 0.\end{aligned} \tag{68}$$

Lemma 5.1. *The solution of* (62) *is given by*

$$X_t = W_t + \int_0^t \left\{\int_0^s L_F(s,u)\,\mathrm{d}W_u + \int_0^s \left(F(s,u) + \int_u^s L_F(s,v)F(v,u)\,\mathrm{d}v\right)\mathrm{d}\tilde{W}_u\right\}\mathrm{d}s. \tag{69}$$

Proof. By (62), the process ξ defined in (50) has the form

$$\mathrm{d}\xi_t = \mathrm{d}X_t + \int_0^t G_F(t,u)\mathrm{d}X_u\,\mathrm{d}t.$$

Using the same argument as in the proof in Section 3.2, we obtain the representation

$$\mathrm{d}X_t = \mathrm{d}\xi_t + \int_0^t L_F(t,u)\mathrm{d}\xi_u\,\mathrm{d}t. \tag{70}$$

Substituting (50) in (70), we obtain (69). □

In the special case $F(t,u)=f(t)g(u)$, we have $G_F(t,s)=f(t)\alpha(s)$, and the solution L_F of (68) is given by

$$L_F(t,s) = -f(t)\alpha(s)\exp\left(-\int_s^t f(v)\alpha(v)\mathrm{d}v\right).$$

Thus, the solution of (64) can be written as

$$\begin{aligned}\mathrm{d}X_t = \mathrm{d}W_t + f(t)&\left(\int_0^t g(u)\exp\left(-\int_u^t f(v)\alpha(v)\mathrm{d}v\right)\mathrm{d}\tilde{W}_u\right.\\ &\left.-\int_0^t \alpha(u)\exp\left(-\int_u^t f(v)\alpha(v)\mathrm{d}v\right)\mathrm{d}W_u\right)\mathrm{d}t.\end{aligned}$$

In particular, the solution of (65) is given by

$$\mathrm{d}X_t = \mathrm{d}W_t + \frac{f(t)}{\Psi'(t)}\left(\int_0^t \Psi'(u)\mathrm{d}\tilde{W}_u - \int_0^t f(u)\Psi(u)\mathrm{d}W_u\right)\mathrm{d}t. \tag{71}$$

We are now going to show that the decomposition (71) of W as the sum

$$W_t = X_t + \int_0^t \frac{f(s)}{\Psi'(s)}\left(\int_0^s f(u)\Psi(u)\mathrm{d}W_u - \int_0^s \Psi'(u)\mathrm{d}\tilde{W}_u\right)\mathrm{d}s$$

can be viewed, after time reversal, as the expression of a Brownian motion in an enlarged Gaussian filtration (see, Jeulin et al., 1985; or Yor (1997)).

We consider an n-dimensional Brownian motion $B_t \equiv (B_t^{(1)},\ldots,B_t^{(n)})$, and we enlarge its filtration with $B(\lambda) \stackrel{\text{def}}{=} \int_0^\infty (\lambda(u),\mathrm{d}B_u)$. With respect to the enlarged filtration, the canonical decomposition of B is given by

$$B_t = \tilde{B}_t + \int_0^t \frac{\lambda(u)\int_u^\infty(\lambda(s),\mathrm{d}B_s)}{\sigma_\lambda^2(u)}\mathrm{d}u, \tag{72}$$

where $\tilde{B}$ is again a Brownian motion and σ_λ is defined as

$$\sigma_\lambda^2(u) = \int_u^\infty |\lambda(t)|^2\mathrm{d}t \equiv \int_u^\infty \left(\sum_{i=1}^n(\lambda_i(t))^2\right)\mathrm{d}t;$$

see Jeulin et al. (1985) or Yor (1997). In order to make sure that formula (72) is meaningful (as a semimartingale decomposition), it is necessary and sufficient that

$$\int_0^t \frac{|\lambda(u)|}{\sigma_\lambda(u)}\mathrm{d}u < \infty.$$

On the other hand, (71) yields

$$B_t^{(1)} = \tilde{B}_t^{(1)} - \int_0^t \frac{f(1-u)}{\Psi'(1-u)} \left(\int_u^1 \Psi'(1-v) \mathrm{d}B_v^{(2)} \right.$$
$$\left. - \int_u^1 f(1-v)\Psi(1-v)\mathrm{d}B_v^{(1)} \right) \mathrm{d}u, \tag{73}$$

with $B_t^{(1)} = W_1 - W_{1-t}$, $\tilde{B}_t^{(1)} = X_1 - X_{1-t}$ and $B_t^{(2)} = \tilde{W}_1 - \tilde{W}_{1-t}$. We want to view the representation (73) as an enlargement formula (72) for a suitable choice of λ. Thus, we would like to find a pair of functions (λ_1, λ_2) such that

$$\begin{aligned} \frac{\lambda_1(u)\lambda_1(s)}{\sigma_\lambda^2(u)} &= \frac{f(1-u)f(1-s)\Psi(1-s)}{\Psi'(1-u)}, \\ \frac{\lambda_1(u)\lambda_2(s)}{\sigma_\lambda^2(u)} &= -\frac{f(1-u)\Psi'(1-s)}{\Psi'(1-u)}. \end{aligned} \tag{74}$$

The problem is now to retrieve λ_1 and λ_2 from the system (74). The solution is given by

$$\lambda_1(s) = cf(1-s)\Psi(1-s),$$
$$\lambda_2(s) = -c\Psi'(1-s),$$

with some nonzero constant c.

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